Constructing new families of transmission irregular graphs

Kexiang Xu[a], Sandi Klavžar[b,c,d]

[a] College of Science, Nanjing University of Aeronautics & Astronautics,
Nanjing, Jiangsu 210016, PR China

[b] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

[c] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

[d] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

kexxu1221@126.com (K. Xu)

sandi.klavzar@fmf.uni-lj.si (S. Klavžar)

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Abstract

The transmission of a vertex v of a graph G is the sum of distances from v to all the other vertices in G. A graph is transmission irregular if all of its vertices have pairwise different transmissions. A starlike tree $T(k_1, \ldots, k_t)$ is a tree obtained by attaching to an isolated vertex t pendant paths of lengths $k_1, \ldots, k_t$, respectively. It is proved that if a starlike tree $T(a, a+1, \ldots, a+k)$, $k \geq 2$, is of odd order, then it is transmission irregular. $T(1, 2, \ldots, \ell)$, $\ell \geq 3$, is transmission irregular if and only if $\ell \notin \{r^2 + 1 : \ r \geq 2\}$. Additional infinite families among the starlike trees and bi-starlike trees are determined. Transmission irregular unicyclic infinite families are also presented, in particular, the line graph of $T(a, a + 1, a + 2)$, $a \geq 2$, is transmission irregular if and only if a is even.

Keywords: graph distance; Wiener complexity; transmission irregular graphs; starlike trees

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1 Introduction

If $G = (V(G), E(G))$ is a graph, we use the notations $n(G) = |V(G)|$ and $m(G) = |E(G)|$, and denote by $d_G(u, v)$ the shortest-path distance between vertices $u, v \in V(G)$. The *transmission* $\mathrm{Tr}_G(v)$ (or Tr(v) for short if the graph G is clear from the context) of a vertex $v \in V(G)$ is the sum of distances from v to the vertices in G, that is,

$$\mathrm{Tr}_G(v) = \sum_{u \in V(G)} d_G(u, v)\,.$$

With this notation we have $W(G) = \frac{1}{2} \sum\limits_{v \in V(G)} \mathrm{Tr}_G(v)$, where $W(G)$ is the famous Wiener index of G. The *Wiener complexity* $C_W(G)$ of a graph G was introduced in [1] (under the name Wiener dimension) as the number of different transmission of vertices in G:

$$C_W(G) = |\{\mathrm{Tr}_G(v) : \ v \in V(G)\}|\,.$$

The Wiener complexity of graphs has been further investigated in [3, 5, 16, 17, 20]. Complexities of related invariants of interest in mathematical chemistry have also been investigated; the complexity of the connective eccentric index in [4, 10], the eccentric complexity in [2], and the complexity of the Szeged index in [6].

The *transmission set* $\mathrm{Tr}(G)$ of G is the set of the transmissions of its vertices, that is, $\mathrm{Tr}(G) = \{\mathrm{Tr}_G(v) : \ v \in V(G)\}$. A graph G is *transmission regular* [19] if all its vertices have the same transmission. In other words, transmission regular graphs are the graphs G with $C_W(G) = 1 = |\mathrm{Tr}(G)|$. On the other extreme, G is *transmission irregular* [5] if all its vertices have pairwise different transmissions, that is, if $C_W(G) = n(G) = |\mathrm{Tr}(G)|$. We note in passing that very recently stepwise transmission irregular graphs were introduced in [15] as the graphs in which the transmissions of any two of its adjacent vertices differ by exactly one.

Now, since almost no graphs are transmission irregular [5], it is of interest to search for families of transmission irregular graphs. For this sake let $t \geq 3$, and let $k_1 \ldots, k_t$ be positive integers. Then a *starlike tree* $T(k_1, \ldots, k_t)$ is a tree obtained by attaching to an isolated vertex t pendant paths of lengths $k_1, \ldots, k_t$, respectively [9, 18]. These pendant paths will be called *k_i-arms*. We may assume without loss of generality throughout the paper that $k_1 \leq \cdots \leq k_t$. In [5] it was proved that $T(1, k_2, k_3)$ is transmission irregular if and only if $k_3 = k_2 + 1$ and $k_2 \notin \{(t^2-1)/2, (t^2-2)/2\}$ for some $t \geq 3$. Al-Yakoob and Stevanović [7] recently extended the latter result by characterizing the starlike trees $T(k_1, k_2, k_3)$ which are transmission irregular, their result will be restated in Theorem 2.1. In the meantime, Dobrynin constructed several families of transmission irregular graphs. In [12] he presented an infinite family of 2-connected transmission irregular graphs, in [14] he followed with an infinite family of 3-connected cubic transmission irregular graphs, while in [13] he discovered an infinite family of transmission irregular trees of even order.

In the rest of this section we recall a few definitions needed and prove some preliminary results. In the first main result of Section 2 we prove that if a starlike tree $T = T(a, a+1, \ldots, a+k)$, $k \geq 2$, is of odd order, then T is transmission irregular. In the second main result of the section we then prove that a starlike tree $T(1, 2, \ldots, \ell)$, $\ell \geq 3$, is transmission irregular if and only if $\ell \notin \{r^2 + 1 : \ r \geq 2\}$. Then, in Section 3, we determine additional infinite families among the starlike trees (broken unit arithmetic starlike trees and extremal starlike trees) and bi-starlike trees. In the subsequent section we turn out attention to unicyclic graph containing C_3. From the two results proved we select the one asserting that the line graph of $T(a, a+1, a+2)$, $a \geq 2$, is transmission irregular if and only if a is even.

1.1 Preliminaries

If k is a positive integer, then $[k] = \{1, \ldots, k\}$ and $[k]_0 = \{0, 1, 2, \ldots, k\}$. The degree of a vertex v of a graph G is denoted by $\deg_G(v)$. A vertex in a tree T of degree at least 3 is called a *branching vertex* in T. The line graph of a graph G is denote by $L(G)$. For an edge $e = uv$ of a graph G, the number of vertices that are closer to u than to v is denoted by $n_u(e|G)$ or n_u for short. Analogously, $n_v(e|G)$ or n_v for short denotes the number of vertices closer to v than to u in G. If A is a set of integers and $i \in \mathbb{Z}$, then $A + i$ denotes the usual coset, that is, $A + i = \{a + i : \ a \in A\}$.

We will make use of the following easy result on the transmission.

Lemma 1.1 ([8]) *If $uv \in E(G)$, then* $\mathrm{Tr}(u) - \mathrm{Tr}(v) = n_v - n_u$.

If T is a tree, then $n_u + n_v = n(T)$ for any edge $uv \in E(T)$. Hence in every tree T there is at most one edge uv for which $n_u = n_v$ holds. Moreover, if such an edge exists, then $n(T)$ must be even. Combining this fact with Lemma 1.1, we have the following result.

Proposition 1.2 *If T is a transmission irregular tree, then T contains no edge uv with $n_u = n_v$.*

Using Lemma 1.1 we also derive the following result.

Proposition 1.3 *If T is a transmission irregular tree, then T contains no two edges $e_1 = xy$ and $e_2 = uv$ with $|n_x - n_y| = |n_u - n_v| = 1$.*

Proof. From Lemma 1.1 we get $\mathrm{Tr}(x) = \mathrm{Tr}(y) + n_y - n_x$. Since $|n_x - n_y| = |n_u - n_v| = 1$, the edges xy and uv must be adjacent in T. We may thus assume without loss of generality that $e_2 = yz$, where $z \neq x$. Then $|n_x - n_y| = |n_z - n_y| = 1$. Since $n_x + n_y = n(T) = n_z + n_y$, we get that $n_x = n_z$. Hence, using Lemma 1.1 again, we get $\mathrm{Tr}(z) = \mathrm{Tr}(y) + n_y - n_z = \mathrm{Tr}(y) + n_y - n_x = \mathrm{Tr}(x)$, contradicting the assumption that T is transmission irregular. $\square$

Proposition 1.4 *Let G be a connected graph with $n(G) = n$ and $v \in V(G)$ of degree $\deg(v) \geq 3$. If $P = uv_1v_2 \cdots v_{x-1}v$ is a pendant path with natural adjacency relation attaching at v, where $\deg(u) = 1$ and $x < \frac{n}{2}$, then $\mathrm{Tr}(v_{x-1}) - \mathrm{Tr}(v) = n - 2x$.*

Proof. By definition, we have $n_u = 1$ and $n_{v_1} = n - 1$, that is, $n_{v_1} - n_u = n - 2$. Similarly, $n_{v_2} - n_{v_1} = n - 4, \ldots, n_{v_{x-1}} - n_{v_{x-2}} = n - 2(x-1)$, and $n_v - n_{v_{x-1}} = n - 2x > 0$. By Lemma 1.1, we get $\mathrm{Tr}(v_{x-1}) - \mathrm{Tr}(v) = n - 2x$. $\square$

We conclude the preliminaries with the following necessary condition for transmission irregular starlike trees.

Proposition 1.5 *If $T(k_1, \ldots, k_t)$ is transmission irregular, then $k_t \leq \sum_{i=1}^{t-1} k_i$.*

Proof. Set $T = T(k_1, \ldots, k_t)$ and suppose on the contrary that $k_t > \sum_{i=1}^{t-1} k_i$. Since $n(T) = 1 + \sum_{i=1}^{t} k_i$, we get that $k_t > \frac{n(T)}{2}$. Let P be the k_t-arm in $T(k_1, \ldots, k_t)$. Based on the parity of $n(T)$, we observe that there exists an edge uv on P with $n_u = n_v$ if $n(T)$ is even, or there are two adjacent edges xy and yz with $|n_x - n_y| = |n_y - n_z| = 1$. By Propositions 1.2 and 1.3, T is not transmission irregular. This contradiction completes the proof. $\square$

2 Unit arithmetic starlike trees

As already mentioned in the introduction, transmission irregular trees $T(1, k_2, k_3)$ were characterized in [5], while in [7] the result was extended to all starlike trees $T(k_1, k_2, k_3)$. We now restate this appealing result to show that the problem is intricate, as well as to be applied later on. Note that its condition $k_3 \leq k_1 + k_2$ is just the case $t = 3$ of Proposition 1.5.

Theorem 2.1 [7, Theorem 2] *$T(k_1,k_2,k_3)$ is transmission irregular if and only if $k_1 < k_2 < k_3$, $k_3 \le k_1+k_2$, and the triplet (k_1,k_2,k_3) does not belong to the set*

$$\bigcup_{1\le i<j} \mathcal{N}_{ij}^{xy} \cup \bigcup_{1\le j<k} \mathcal{N}_{jk}^{yz} \cup \bigcup_{1\le i<k} \mathcal{N}_{ik}^{xz}\,,$$

where

$$\begin{aligned}
\mathcal{N}_{ij}^{xy} &= \Big\{\Big(k_1, k_1+(j-i)\Big(1+\frac{p}{\gcd(i+j,j-i)}\Big), \frac{p(i+j)}{\gcd(i+j,j-i)}\Big) : \\
&\quad i\le k_1,\ \frac{(k_1+j-i)\gcd(i+j,j-i)}{2i} \le p\Big\}\,,\\
\mathcal{N}_{jk}^{yz} &= \Big\{\Big(\frac{p(j+k)}{\gcd(j+k,k-j)}, k_2, k_2+(k-j)\Big(1+\frac{p}{\gcd(j+k,k-j)}\Big)\Big) : \\
&\quad \max\Big(j, \frac{j+k}{\gcd(j+k,k-j)}\Big) \le k_2,\ 1\le p\le \frac{k_2\gcd(j+k,k-j)}{j+k}\Big\}\,,\\
\mathcal{N}_{ik}^{xz} &= \Big\{\Big(k_1, \frac{p(i+k)}{\gcd(i+k,k-i)}, k_1+(k-i)\Big(\frac{p}{\gcd(i+k,k-i)}\Big)\Big) : \\
&\quad i\le k_1,\ \frac{k_1\gcd(i+k,k-i)}{i+k} \le p \le \frac{(k_1+k-i)\gcd(i+k,k-i)}{2i}\Big\}\,.
\end{aligned}$$

As pointed out by Al-Yokoob and Stevanović, the proof method used to prove Theorem 2.1 could in principle be applied also to starlike trees with more than three arms. However, the number of sets of parameter values to be avoided grows quadratically with the number of branches, so possible formulations of such results (as well as their proofs) would be extremely long and consequently useless. Moreover, computational results (see [7, Table 1]) indicate that the number of transmission irregular starlike trees rapidly decreases with the number of branches. Nevertheless we will construct in this section infinite families of transmission irregular starlike trees with an arbitrary number of arms.

We say that a starlike tree $T(k_1,\ldots,k_t)$ is *arithmetic* if $k_{i+1}-k_i$ is a constant, or *unit arithmetic* if $k_{i+1}-k_i = 1$, for $i\in[t-1]$. The main result of this section reads as follows.

Theorem 2.2 *If T is a unit arithmetic starlike tree of odd order, then T is transmission irregular.*

Proof. Let $T = T(a, a+1, \ldots, a+k)$, $k\ge 2$, and let v be the vertex of T with degree $k+1$. For $p\in[a+k]$ define the sets B_p as follows:

$$B_p = \begin{cases} \{ps + p(p-1) + 2pi :\ i\in[k+1]\}; & p\in[a],\\[2ex] \{ps + p(p-1) + 2pi :\ i\in[(k+1)-(p-a)]\}; & p\in[a+k]\setminus[a]\,, \end{cases}$$

where $s = (k-1)(a + \frac{k}{2} - 1) - 2$.

Claim A: $\mathrm{Tr}(T) \setminus \{\mathrm{Tr}(v)\} = \bigcup_{p=1}^{a+k} \Big(B_p + (\mathrm{Tr}(v) + s + 2)\Big)$.

By the structure of T we see that $n(T) = 1+a+(a+1)+\cdots+(a+k) = (k+1)(a+\frac{k}{2})+1$. Since the distance between v and the leaf on the longest arm is is $a+k$, we have $\mathrm{Tr}(v') - \mathrm{Tr}(v) = (k+1)(a+\frac{k}{2}) + 1 - 2(a+k) = (k-1)(a+\frac{k}{2}-1)$ by Proposition 1.4, where v' is the neighbor of v lying on the longest arm of T. From Lemma 1.1, the transmission of the vertex w is $ps + p(p-1) + 2pi$ if w is on the $(a+k+1-i)$-arm of T with $d(w,v) = p$ and $i \in [k+1]$ for $p \in [a]$ or $i \in [(k+1)-(p-a)]$ for $p \in [a+k] \setminus [a]$ where $s = (k-1)(a+\frac{k}{2}-1) - 2$. Claim A now follows from the definition of B_p. $(\square)$

By the assumption, the order of T, $n(T) = (k+1)(a+\frac{k}{2}) + 1$, is odd. Note that $\mathrm{Tr}(u) \neq \mathrm{Tr}(v)$ for any vertex $u \in V(T) \setminus \{v\}$. Then it suffices to prove that

$$|\{\mathrm{Tr}(u) : u \in V(T) \setminus \{v\}\}| = n(T) - 1.$$

Note that $|A + a| = |A|$ for any set A. By Claim A and the definition of B_p, it suffices to prove that $|\bigcup_{p=1}^{a+k} B_p| = n(T) - 1$, that is, the sets B_p, $p \in [a+k]$, are pairwise disjoint. Recall that $s = (k-1)(a+\frac{k}{2}-1) - 2$ and note that s is odd. Then B_p consists of increasingly odd numbers in terms of i if p is odd, or of increasingly even numbers in terms of i if p is even. Set $B^{(1)} = \{B_p : p \in [a+k] \text{ is odd}\}$ and $B^{(2)} = \{B_p : p \in [a+k] \text{ is even}\}$. Since $B^{(1)} \cap B^{(2)} = \emptyset$, it suffices to prove that $B_p \cap B_{p+2t} = \emptyset$ for any subset $\{p, p+2t\} \subseteq [a+k]$. Next we calculate the value of $\min B_{p+2} - \max B_p$. If $\{p, p+2\} \subseteq [a]$, we have

$$\begin{aligned}
\min B_{p+2} - \max B_p &= (p+2)s + (p+2)(p+1) + 2(p+2) \\
&\quad -ps - p(p-1) - 2p(k+1) \\
&= 2s + 6p + 6 - 2p(k+1) \\
&= (k-1)(2a+k-2) + 6p + 2 - 2p(k+1) \\
&= (k-1)(2a - 2p + k - 2) + 2p + 2 \\
&> 0\,.
\end{aligned}$$

If $p \in [a]$ and $p+2 \in [a+k] \setminus [a]$, similarly as above, we can get $\min B_{p+2} - \max B_p > 0$. While $\{p, p+2\} \subseteq [a+k] \setminus [a]$, we have

$$\begin{aligned}
\min B_{p+2} - \max B_p &= (p+2)s + (p+2)(p+1) + 2(p+2) \\
&\quad -ps - p(p-1) - 2p[(k+1) - (p-a)] \\
&= 2s + 6p + 6 - 2p[(k+1) - (p-a)] \\
&= (k-1)(2a+k-2) + 6p + 2 - 2p(k+1) + 2p(p-a) \\
&= (k-1)\Big[k-2-2(p-a)\Big] - 4p + 6p + 2 + 2p(p-a) \\
&= (k-1)(k-2) + 2(p-a)(p-k+1) + 2p + 2.
\end{aligned}$$

Set $x = (k-1)(k-2)+2(p-a)(p-k+1)+2p+2$. Note that $k \geq 3$ and $a < p \leq a+k-2$. If $a \geq k$ or $a < k \leq p$, then $x > 0$ holds clearly. Now we consider the last case $a < p < k$. In this case, since $k - p \geq 1$, we have

$$\begin{aligned}
x &= (k-1)(k-2) - 2(k-p)(p-a) + 4p - 2a + 2 \\
&= k^2 - 3k + 2 - 2(kp - p^2 - ak + ap) + 4p - 2a + 2 \\
&= (k-p)^2 + p^2 + 2a(k-p) + 4p - 2a - 3k + 4 \\
&\geq 2p(k-p) + 2a(k-p) - 3(k-p) + p - 2a + 4 \\
&= (2p + 2a - 3)(k-p) + p + 4 - 2a \\
&\geq 2p + 2a - 3 + p + 4 - 2a \\
&= 3p + 1 \\
&> 0.
\end{aligned}$$

Thus the sets B_p, $p \in [a+k]$, are pairwise disjoint, completing the proof. □

It can be routinely checked that $(k+1)(a+\frac{k}{2})+1$ is odd if and only if $k \equiv 3 (\mathrm{mod}\ 4)$ or $k + 2a \equiv 0 (\mathrm{mod}\ 4)$. Therefore, we have the following consequence.

Corollary 2.3 *If $k \equiv 3 (\mathrm{mod}\ 4)$ or $k + 2a \equiv 0 (\mathrm{mod}\ 4)$, then $T(a, a+1, \ldots, a+k)$ is transmission irregular.*

From Corollary 2.3 we can obtain some special transmission irregularity starlike trees. For instance, $T(a, a+1, a+2, \ldots, a+k)$ is transmission irregular when $k \equiv 2 (\mathrm{mod}\ 4)$ and a is odd. This fact for $k = 2$ and odd a enlarges the set of transmission irregular starlike trees included in [5, Table 1]. Moreover, we also have the following characterization for this case.

Corollary 2.4 *$T(a, a+1, a+2)$ is transmission irregular if and only if a is odd.*

Proof. By the above we only need to prove that $T(a, a+1, a+2)$ is not transmission irregular if a is even. Assume that $a = 2t$ with $t \geq 1$. Setting $a = 2t$ and $k = 2$ in B_p, we have

$$B_p = \begin{cases} \{ps + p(p-1) + 2pi : i \in [3]\}; & p \in [2t], \\ \\ \{ps + p(p-1) + 2pi : i \in [2t+3-p]\}; & p \in \{2t+1, 2t+2\}, \end{cases}$$

where $s = 2t - 2$. Thus $\min B_{t+1} = (t+1)(s+t+2) = \max B_t$, which implies that $T(a, a+1, a+2)$ is not transmission irregular for $a = 2t$ with $t \geq 1$. $\square$

Corollary 2.4 can also be deduced from Theorem 2.1. In the theorem, set $k_1 = a$, $k_2 = a+1$, and $k_3 = a+2$. Then it is easily seen that the sets $\mathcal{N}_{ij}^{xy}$ and $\mathcal{N}_{jk}^{yz}$ are empty. For the set $\mathcal{N}_{ik}^{xz}$ we get that $k - i = 1$ and $p = \gcd(i+k, k-i) = 1$ must hold. For the second coordinate we have $i + k = a + 1$, from which we get $2i = a$. Hence $(a, a+1, a+2) \in \mathcal{N}_{ik}^{xz}$ if and only if a is even which, by Theorem 2.1, in turn implies that $T(a, a+1, a+2)$ is transmission irregular if and only if a is odd.

In the next result we provide a complete characterization of the transmission irregularity of arithmetic starlike trees with $k_1 = 1$.

Theorem 2.5 *Let $T = T(1, 2, \ldots, \ell)$ with $\ell \geq 3$. Then T is transmission irregular if and only if $\ell \notin \{r^2 + 1 : \ r \geq 2\}$.*

Proof. Assume that v is the vertex of maximum degree in T. Setting $a = 1$ and $k = \ell - 1$ in Claim A, we have $\mathrm{Tr}(T) \setminus \{\mathrm{Tr}(v)\} = \bigcup\limits_{p=1}^{\ell} \Big(B_p + (\mathrm{Tr}(v) + s + 2)\Big)$ where $B_p = \{ps + p(p-1) + 2pi : i \in [\ell + 1 - p]\}$ with $s = (k-1)(a + \frac{k}{2} - 1) - 2 = \frac{\ell(\ell-3)}{2} - 1$ for $p \in [\ell]$. Clearly, B_p is a set of increasing elements in terms of i. For convenience, we write $B_{p,i} = ps + p(p-1) + 2pi$. Observe that T is transmission irregular if and only if $|\bigcup\limits_{p=1}^{\ell} B_p| = n(T) - 1$, that is, the sets B_p, $p \in [\ell]$, are pairwise disjoint. Let $B'_p = B_p \setminus \{ps + p(p-1) + 2p(\ell + 1 - p)\}$. Then we have

$$\begin{aligned} \min B_p - \max B'_{p-1} &= \frac{\ell(\ell-3)}{2} + p^2 - (p-1)(p-3) - 2(p-1)(\ell+1-p) \\ &= \frac{\ell(\ell-3)}{2} + 4p - 3 - 2(p-1)(\ell+1-p) \\ &= \frac{\ell(\ell-3)}{2} + 2p^2 - 2\ell p + 2\ell - 1 \\ &= 2\Big(p - \frac{\ell}{2}\Big)^2 + \frac{\ell-2}{2} \\ &> 0, \end{aligned}$$

that is, $\min B_p$ is larger than the second largest element in B_{p-1} for any $p \in [\ell] \setminus \{1\}$. Note that

$$\begin{aligned}
\min B_p - \max B_{p-1} &= \frac{\ell(\ell-3)}{2} + p^2 - (p-1)(p-3) - 2(p-1)(\ell+2-p) \\
&= \frac{\ell(\ell-3)}{2} + 2p^2 - 2(\ell+1)p + 2\ell + 1 \\
&= 2\Big(p - \frac{\ell+1}{2}\Big)^2 - \frac{\ell-1}{2} \\
&\geq 0
\end{aligned}$$

for any $p \in [1, \frac{\ell+1-\sqrt{\ell-1}}{2}] \cup [\frac{\ell+1+\sqrt{\ell-1}}{2}, \ell]$. Moreover,

$$\begin{aligned}
\max B_{p-1} - \min B_p &= \frac{\ell-1}{2} - 2\Big(p - \frac{\ell+1}{2}\Big)^2 \\
&\geq 0
\end{aligned}$$

for any $p \in [\frac{\ell+1-\sqrt{\ell-1}}{2}, \frac{\ell+1+\sqrt{\ell-1}}{2}]$. So $\max B_{p-1} \leq \min B_p \leq \min B'_{p-1}$ for $p \in [\ell] \setminus \{1\}$. In view of Theorem 2.2, we observe that T is transmission irregular if and only if $n(T)$ is odd, or otherwise $\ell + 1 + \sqrt{\ell - 1}$ is not even. Note that $n(T) = \frac{\ell(\ell+1)}{2} + 1$ is odd if and only if $\ell \equiv j (\mathrm{mod}\ 4)$ with $j \in \{0, 3\}$.

We have thus proved that T is not transmission irregular if and only if $\ell \equiv j (\mathrm{mod}\ 4)$ with $j \in \{1, 2\}$ and $\ell + 1 + \sqrt{\ell - 1}$ is even. Suppose that $\sqrt{\ell - 1} = r \in \mathbb{Z}^+$. Then $\ell = r^2 + 1$ and hence $\ell + 1 + \sqrt{\ell - 1} = (r^2 + 1) + 1 + r = r(r + 1) + 2$, which is even. If $r = 2k$, then $\ell = 4k^2 + 1$, so $\ell \equiv 1 (\mathrm{mod}\ 4)$. And if $r = 2k + 1$, then $\ell = 4k(k + 1) + 2$, so $\ell \equiv 2 (\mathrm{mod}\ 4)$. We conclude that T is not transmission irregular if and only if $\ell \in \{r^2 + 1 : \ r \geq 2\}$. $\square$

To conclude the section we give a negative result by proving the a certain family of arithmetic starlike trees is not transmission irregular.

Theorem 2.6 *If* $\frac{2(a-3)}{3} \leq k \leq 2a + 2$ *and* $k + 2a \equiv 2 (\mathrm{mod}\ 4)$*, then* $T(a, a+1, a+2, \ldots, a+k)$ *is not transmission irregular.*

Proof. Assume that $k + 2a = 4x + 2$. Then $a + \frac{k}{2} - 1 = 2x$. Since $\frac{2(a-3)}{3} \leq k \leq 2a + 2$, we have $x \leq \min\{a, k + 1\}$. Let v be the vertex of maximum degree in $T(a, a+1, a+2, \ldots, a+k)$. As stated in the proof of Theorem 2.2, we have $\max B_x = xs + x(x-1) + 2x(k+1)$ and $\min B_{x+1} = (x+1)s + x(x+1) + 2(x+1)$ with $s = (k-1)(a + \frac{k}{2} - 1) - 2$. A straightforward calculation shows that $\max B_x = \min B_{x+1}$, which implies that $C_W(T) \leq n(T) - 1$. $\square$

3 More (bi-)starlike transmission irregular trees

3.1 Broken unit arithmetic starlike trees

A starlike tree is *broken unit arithmetic* if its arm length set is obtained from a unit arithmetic sequence by removing some consecutive elements of the sequence. If $a_1, \ldots, a_k$ is a unit arithmetic sequence in which all elements from the open interval (a_i, a_j) were removed, where $1 \le i < j-1 \le k-1$, then the broken unit arithmetic starlike tree corresponding to this new sequence will be denoted by $T[a_1, a_i; a_j, a_k]$. Below we present the transmission irregularity of a special class of broken arithmetic starlike trees $T[a, a+k-2; a+k, a+k+1]$.

Theorem 3.1 *Let $T = T[a, a+k-2; a+k, a+k+1]$ with $k \ge 2$. If $k \equiv 3 (\bmod\ 4)$ or $k + 2a \equiv 0 (\bmod\ 4)$, then T is transmission irregular.*

Proof. Assume that $T_0 = T(a, a+1, \ldots, a+k)$ with u_0 and v_0 being the pendant vertices of the $(a+k)$-arm and the $(a+k-1)$-arm of T_0, respectively. Note that u_0 and v_0 are the diametrical vertices of T_0. Then T can be obtained from T_0 by adding pendant vertices u and v, and edges uu_0 and vv_0. Let $w \in V(T)$ be the branching vertex of T and let B_p and s be defined as that in the proof of Theorem 2.2. For $p \in [a+k]$ we define a new set B'_p which consists of the first two elements of B_p, that is, $B'_p = \{ps + p(p-1) + 2pi : \ i \in [2]\}$, and define in addition the sets B''_p by

$$B''_p = \begin{cases} \{ps + p(p-1) + 2(p+1)i : \ i \in [k+1]\}; & p \in [a], \\ \\ \{ps + p(p-1) + 2(p+1)i : \ i \in [(k+1) - (p-a)]\}; & p \in [a+k-2] \setminus [a]\,, \end{cases}$$

with $s = (k-1)(a + \frac{k}{2} - 1) - 2$. Let $B^*_p = B'_p \cup B''_p$ for $p \in [a+k-2]$ and set $B^*_{a+k-1} = B'_{a+k-1}$ for consistency. The transmissions of vertices not on the diametrical path of T form the set $\bigcup\limits_{p=1}^{a+k-2} (B''_p + \mathrm{Tr}_T(w))$, those of vertices but u and w on the diametrical path of T is just $\bigcup\limits_{p=1}^{a+k-1} (B'_p + \mathrm{Tr}_T(w))$. Let $B^*_{a+k} = \{\mathrm{Tr}_T(u_0), \mathrm{Tr}_T(v)\}$ and $B^*_{a+k+1} = \{\mathrm{Tr}_T(u)\}$. Note that $n(T) = n(T_0) + 2$ is odd from the assumption. Then

$$\mathrm{Tr}(T) \setminus \{\mathrm{Tr}_T(w)\} = \bigcup_{p=1}^{a+k+1} B^*_p\,.$$

Moreover, $\mathrm{Tr}_T(z) - \mathrm{Tr}_T(w) = \mathrm{Tr}_{T_0}(z) - \mathrm{Tr}_{T_0}(w)$ for $z \in \{u_0, v_0\}$. By Lemma 1.1, we have $\mathrm{Tr}_T(u) = \mathrm{Tr}_T(u_0) + n(T) - 2$, and $\mathrm{Tr}_T(v) = \mathrm{Tr}_T(v_0) + n(T) - 2$. From the proof of

Theorem 2.2, we know that $\mathrm{Tr}_{T_0}(u_0)$ and $\mathrm{Tr}_{T_0}(v_0)$ are maximum and second maximum transmissions in $\mathrm{Tr}(T_0)$, so are $\mathrm{Tr}_T(u)$ and $\mathrm{Tr}_T(v)$ in $\mathrm{Tr}(T)$. Now we only need to prove that the sets from the union $\bigcup\limits_{p=1}^{a+k+1} B_p^*$ are pairwise disjoint. By a similar reasoning as that in the proof of Theorem 2.2, it suffices to prove that $\min B_{p+2}^* - \max B_p^* > 0$ for any $\{p, p+2\} \subseteq [a+k]$. Let $x = \min B_{p+2}^* - \max B_p^*$. For any $\{p, p+2\} \subseteq [a]$, we have

$$\begin{aligned} x &= (k-1)(2a-2p+k-2)+2p+2-2(k+1) \\ &= (k-1)[2a-2(p+2)+k]+2p-2 \\ &\geq k(k-1)+2p-2 > 0\,. \end{aligned}$$

Similarly, we have $x > 0$ if $p \in [a]$ and $p+2 \in [a+k]\backslash[a]$. For any $\{p, p+2\} \subseteq [a+k]\backslash[a]$, we have

$$\begin{aligned} x &= (k-1)(k-2)+2(p-a)(p-k+1)+2p+2-2[(k+1)-(p-a)] \\ &= (k-1)(k-4)+2(p-a)(p-k+2)+2p-2. \end{aligned}$$

If $a \geq k$ or $a < k \leq p$, then $x > 0$ holds. If $a < p < k$, we have

$$\begin{aligned} x &= (k-p)^2+p^2+(2a-5)(k-p)+p-4a+2 \\ &\geq (2p+2a-5)(k-p)+p-4a \\ &\geq 3p-2a-3 \\ &\geq 3a+3-2a-3 \\ &= a > 0, \end{aligned}$$

completing the proof. $\square$

3.2 Extremal starlike trees

In view of Proposition 1.5 we say that a starlike tree $T(k_1, \ldots, k_t)$ is *extremal* if $k_t = \sum_{i=1}^{t-1} k_i$ holds. In this section we construct some transmission irregular extremal starlike trees. Similarly as in the proof of Theorem 2.2, for positive integers a and k set where $h = (k-1)(2a+k)+2a-1$ and define the sets D_p, $p \in [a+k]$, as follows:

$$D_p = \begin{cases} \{ph + p(p-1) + 2pi : i \in [k+1]\}; & p \in [a], \\[2ex] \{ph + p(p-1) + 2pi : i \in [(k+1)-(p-a)]\}; & p \in [a+k] \setminus [a]. \end{cases}$$

Mimicking the proof of Theorem 2.2 we can prove the following result, hence its proof is omitted.

Lemma 3.2 *Let a and k be positive integers, and let D_p and h be defined as above. Then the sets D_p, $p \in [a+k]$, are pairwise disjoint.*

With Lemma 3.2 in hand we can find the announced transmission irregular extremal starlike trees.

Theorem 3.3 *Let $T = T(a, a+1, \dots, a+k, (a+\frac{k}{2})(k+1))$, and let $D = \bigcup_{p=1}^{a+k} D_p$, where the sets D_p are defined as above. If for every $d \in D$, the number d is not a square number from the interval $\left[k(2a+k-1)+1, (a+\frac{k}{2})^2(k+1)^2\right]$, then T is transmission irregular.*

Proof. Note that $n(T) = (2a+k)(k+1)+1$. Assume that v is the vertex with maximum degree in T and $\mathrm{Tr}(v) = y$. By Proposition 1.4, we have $\mathrm{Tr}(v_1) = y+1$ where v_1 lies on the longest pendant path in T with $d_T(v, v_1) = 1$. By Lemma 1.1, the set of transmissions is just $\{1, 4, 9, \dots, (a+\frac{k}{2})^2(k+1)^2\} + y$ of vertices on the longest pendant in T. Note that $h = (k-1)(2a+k)+2a-1$. From the structure of T, we observe that the set of transmissions is just $D+y$ of vertices in T different from v and not lying the longest arm. From the assumption with Lemma 3.2, our result follows. $\square$

Taking $k = 1$ in Theorem 3.3, we have the following result.

Corollary 3.4 *Let $T = T(a, a+1, 2a+1)$. If d is not a square number in the interval $\left[2a+1, (2a+1)^2\right]$ for any d in $\{p(2a-1)+p^2 : \ p \in [a+1]\}$ or $\{p(2a-1)+p^2+2p : p \in [a]\}$, then T is transmission irregular.*

3.3 Bi-starlike trees

A tree T is a *bi-starlike tree* if T contains exactly two vertices of degrees at least 3. Note that some special bi-starlike trees and starlike trees are used [11] as extremal ones with respect to two novel distance-based graphic invariants. The length of the induced path connecting these two vertices of degrees at least 3 is called the *shoulder width* of a bi-starlike tree and the path connecting these two vertices of degrees at least 3 is called the *shoulder path* in this bi-starlike tree. Denote by $BT^{(k_i)}(k_1, k_2, \dots, k_t)$ a bi-starlike tree obtained from two copies, say T_1 and T_2, of starlike tree $T(k_1, k_2, \dots, k_t)$ by identifying the branching vertex of T_1 with the leaf on the k_i-arm of T_2. Note that the shoulder width of $BT^{(k_i)}(k_1, k_2, \dots, k_t)$ is k_i.

Remark 3.5 *It can be routinely checked that $BT^{(2a+1)}(a, a+1, 2a+1)$ is transmission irregular for $a \in \{2, 3, 5\}$ but not transmission irregular if $a = 6$.*

From Remark 3.5, it seems a bit difficult to construct transmission irregular bi-starlike trees with long shoulder widths. But the case is different when the shoulder width is short. Although there are some unit arithmetic starlike trees which are not transmission irregular, we can construct transmission irregular bi-starlike trees with shoulder width 1 using unit arithmetic starlike trees regardless of whether they are transmission irregular or not. Denote by $BS^*(a, a+1, a+2, \dots, a+k)$ a bi-starlike tree obtained by connecting two vertices of degree $k+1$ of two copies of $T(a, a+1, a+2, \dots, a+k)$ and attaching a pendant vertex to one vertex of degree $k+2$.

Theorem 3.6 *Let $T^* = BS^*(a, a+1, a+2, \dots, a+k)$ with $a > 1$. Then T^* is transmission irregular.*

Proof. From the structure of T^*, we have $n(T^*) = (k+1)(2a+k)+3$. Assume that $v, v' \in V(T^*)$ with $\deg_{T^*}(v) = k+3$, $\mathrm{Tr}(v) = x$ and $\deg_{T^*}(v') = k+2$. Then $vv' \in E(T)$. By Lemma 1.1, we have $\mathrm{Tr}(v') = x+1$. Let $T^* - vv' = T \cup T'$ where $v \in V(T)$ and $v' \in V(T')$. Setting $t = \binom{k}{2} + ak + 1$, we have $n(T^*) = 2t + 1 + 2(a+k)$. Let v_0 be the leaf adjacent to v in T of T^*. Then $\mathrm{Tr}(v_0) = x + 2t + 2(a+k) - 1$. Now we define a set A_i as follows:

$$A_i = \begin{cases} \{2it + i^2 + 2ij : j \in [k]_0\}; & i \in [a], \\ \\ \{2it + i^2 + 2ij : j \in [k+a-i]_0\}; & p \in [a+k] \setminus [a]. \end{cases}$$

By Proposition 1.4, the transmissions of vertices in T adjacent to v form the set $\Big(\{2t + 2(a+k) - 1\} \cup A_1\Big) + x$ and the transmissions of vertices in T with distance i to v form the set $A_i + x$ for any $i \in [a+k] \setminus \{1\}$. Moreover, $\mathrm{Tr}(u') = \mathrm{Tr}(u) + 1$ for any corresponding vertex u' in T' to u in T of T^*. Note that $|D| = |D + d|$ for any set D and any number d. Then it suffices to prove that the sets $A_1^* = A_1 \cup \{2t + 2(a+k) - 1\}$ and A_i, $i \in \{2, \dots a+k\}$, are pairwise disjoint.

Since $a > 1$, A_1^* is pairwise disjoint. Moreover, by the definition of A_i, we have $\min\limits_{i \in [a+k] \setminus \{1\}} \min A_i = 4t + 4 > 2t + 2(a+k) - 1$ for $a > 1$. Then we only need to prove that A_i, $i \in \{2, \dots a+k\}$, are pairwise disjoint. Note that A_i consists of increasing odd numbers in terms of i if i is odd and vice versa. For any $\{i, i+2\} \subseteq [a]$, we have

$$\begin{aligned} \min A_{i+2} - \max A_i &= 2(i+2)t + (i+2)^2 - 2it - i^2 - 2ik \\ &= 4t + 4i + 4 - 2ik \\ &\geq 2k(2a + k - 1) - 2a(k-2) + 4 \\ &> 0. \end{aligned}$$

From the fact that $\max A_i = 2it + i^2 + 2i(k+a-i) < 2it + i^2 + 2ik$ for $i \in [a+k] \setminus [a]$, our results follows immediately. $\square$

4 Cycle-containing graphs

In this section we will construct some cycle-containing graphs with transmission irregularity. Denote by $C_3(k_1; k_2, k_3; k_4, k_5)$ a graph obtained from a triangle C_3 by attaching at one vertex of C_3 a pendant path of length k_1, at another vertex of C_3 pendant paths of lengths k_2 and k_3, respectively, and at the third vertex pendant paths of lengths k_4 and k_5, respectively.

Proposition 4.1 *Let $k \geq 3$ and $G = C_3(1; 1, k; 2, k)$. Let $A_0 = \{k+9, 2k+11, 2k+14, 3k+14, 4k+16\}$, $A_1 = A+1$, $A = A_0 \cup A_1$, and $B = \{i^2 : \ i \in [k+3] \setminus [2]\}$. If $A \cap B = \emptyset$, then G is transmission irregular.*

Proof. Let w be the unique vertex of degree 3 in G, and let u and v be the two vertices of degree 4 in G. From the structure of G, there is a pendant vertex w' attached at w and there exist a pendant vertex u' and a pendant path $P_u := uu_1u_2\ldots u_{k-1}u_k$ attached at u, two pendant paths $P' := vv'v''$ and $P_v := vv_1v_2\ldots v_{k-1}v_k$ attached at v in G. Note that $n(G) = 2k+7$. By the structure of G, we have $\mathrm{Tr}(w) = k^2 + 3k + 10$, $\mathrm{Tr}(u) = (k+1)^2 + 9$, and $\mathrm{Tr}(v) = (k+1)^2 + 8$. By Proposition 1.4 and Lemma 1.1, we observe that $\mathrm{Tr}(w') = k^2 + 5k + 15$, $\mathrm{Tr}(u') = (k+1)^2 + 2k + 14$, $\mathrm{Tr}(v') = (k+1)^2 + 2k + 11$, $\mathrm{Tr}(v'') = (k+1)^2 + 4k + 16$, the set of vertices on P_u including u is $\{(k+1)^2 + j^2 : j \in [k+3] \setminus [2]\}$ and the set of transmissions of vertices on P_v including v is $\{k^2 + 2k + j^2 : j \in [k+3] \setminus [2]\}$. Therefore, we have

$$\mathrm{Tr}(G) = D \bigcup (B + (k+1)^2) \bigcup (B + (k^2 + 2k)),$$

where $D = \{k+9, 2k+11, 2k+14, 3k+14, 4k+16\} + (k+1)^2$. Thus our result follows from the assumption. $\square$

Next we give a complete characterization of transmission irregularity of the line graph $L(T)$ of $T = T(a, a+1, a+2)$.

Theorem 4.2 *Let $T = T(a, a+1, a+2)$ with $a \geq 2$. Then $L(T)$ is transmission irregular if and only if a is even.*

Proof. Note that $L(T) = C_3(a-1, a, a+1)$ of order $3a+3$. Assume that three vertices of degree 3 in $L(T)$ are u, v, and w at which the attached pendant paths are of lengths $a-1$, a, and $a+1$, respectively. Then $\mathrm{Tr}(u) = \frac{a(3a+7)}{2} + 4$, $\mathrm{Tr}(v) = \frac{a(3a+7)}{2} + 3$, and $\mathrm{Tr}(w) = \frac{a(3a+7)}{2} + 2$. From Proposition 1.4 and Lemma 1.1, the transmissions of the vertices on the $(a-1)$-, a-, and $(a+1)$-arms not including u, v, and w are $A_u + \frac{a(3a+7)}{2}$, $A_v + \frac{a(3a+7)}{2}$ and $A_w + \frac{a(3a+7)}{2}$, where $A_u = \{pa + (p+2)^2 : p \in [a-1]\}$, $A_v = \{pa + (p+1)^2 + 2 : p \in [a]\}$ and $A_w = \{pa + p^2 + 2 : p \in [a+1]\}$. Let $A = \{2, 3, 4\} \cup A_u \cup A_v \cup A_w$. Then $\mathrm{Tr}(L(T)) = A + \frac{a(3a+7)}{2}$. Thus $L(T)$ is transmission irregular if and only if the sets A_u, A_v, and A_w are pairwise disjoint. Since each of A_u, A_v, and A_w consists of increasing numbers in terms of p, the only possible equal numbers in these three sets can happen if we have the following equality:

$$pa + (p+2)^2 = (p+1)a + (p+1)^2 + 2,$$

which implies $2p + 1 = a$. We conclude that the sets A_u, A_v, and A_w are pairwise disjoint if and only if a is even. $\square$

From Theorem 4.2, we conclude that $T(a, a+1, a+2)$ is transmission irregular if and only if $LT(a, a+1, a+2)$ is not transmission irregular. This interesting fact leads to the following problem.

Problem 4.3 *Investigate the correlation between the transmission irregularity of (s-tarlike) trees with that of their line graphs.*

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